

# Lattice structure of torsion classes for hereditary artin algebras.

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**Abstract:** Let  $\Lambda$  be a connected hereditary artin algebra. We show that the set of functorially finite torsion classes of  $\Lambda$ -modules is a lattice if and only if  $\Lambda$  is either representation-finite (thus a Dynkin algebra) or  $\Lambda$  has only two simple modules. For the case of  $\Lambda$  being the path algebra of a quiver, this result has recently been established by Iyama-Reiten-Thomas-Todorov and our proof follows closely their considerations.

Let  $\Lambda$  be a connected hereditary artin algebra. The modules considered here are left  $\Lambda$ -modules of finite length,  $\text{mod } \Lambda$  denotes the corresponding category. The subcategories of  $\text{mod } \Lambda$  we deal with are always assumed to be closed under direct sums and direct summands (in particular closed under isomorphisms). In this setting, a subcategory is a *torsion class* (the class of torsion modules for what is called a torsion pair or a torsion theory) provided it is closed under factor modules and extensions. The torsion classes form a partially ordered set with respect to inclusion, it will be denoted by  $\text{tors } \Lambda$ . This poset clearly is a lattice (even a complete lattice). Auslander and Smalø have pointed out that a torsion class  $\mathcal{C}$  in  $\text{mod } \Lambda$  is functorially finite if and only if it has a cover (a *cover* for  $\mathcal{C}$  is a module  $C$  such that  $\mathcal{C}$  is the set of modules generated by  $C$ ), we denote by  $\text{f-tors } \Lambda$  the set of functorially finite torsion classes in  $\text{mod } \Lambda$ .

In a recent paper [IRTT], Iyama, Reiten, Thomas and Todorov have discussed the question whether also the poset  $\text{f-tors } \Lambda$  (with the inclusion order) is a lattice.

**Theorem.** *The poset  $\text{f-tors } \Lambda$  is a lattice if and only if  $\Lambda$  is representation finite or  $\Lambda$  has precisely two simple modules.*

Iyama, Reiten, Thomas, Todorov have shown this in the special case when  $\Lambda$  is a  $k$ -algebra with  $k$  an algebraically closed field (so that  $\Lambda$  is Morita equivalent to the path algebra of a quiver). The aim of this note is to provide a proof in general. We follow closely the strategy of the paper [IRTT] and we will use Remark 1.13 of [IRTT] which asserts that a meet or a join of two elements  $\mathcal{C}_1, \mathcal{C}_2$  in  $\text{f-tors } \Lambda$  exists if and only if the meet or the join of  $\mathcal{C}_1, \mathcal{C}_2$  formed in  $\text{tors } \Lambda$  belongs to  $\text{f-tors } \Lambda$ , respectively.

## 1. Normalization.

Let  $\mathcal{X}$  be a class of modules. We denote by  $\text{add}(\mathcal{X})$  the modules which are direct summands of direct sums of modules in  $\mathcal{X}$ . A module  $M$  is *generated* by  $\mathcal{X}$  provided  $M$  is a factor module of a module in  $\text{add}(\mathcal{X})$ , and  $M$  is *cogenerated* by  $\mathcal{X}$  provided  $M$  is a submodule of a module in  $\text{add}(\mathcal{X})$ . The subcategory of all modules generated by  $\mathcal{X}$  is denoted by  $\mathcal{G}(\mathcal{X})$ . In case  $\mathcal{X} = \{X\}$  or  $\mathcal{X} = \text{add } X$ , we write  $\mathcal{G}(X)$  instead of  $\mathcal{G}(\mathcal{X})$ , and

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use the same convention in similar situations. We write  $\mathcal{T}(X)$  for the smallest torsion class containing the module  $X$  (it is the intersection of all torsion classes containing  $X$ , and it can be constructed as the closure of  $\{X\}$  using factor modules and extensions).

Since  $\Lambda$  is assumed to be hereditary, we write  $\text{Ext}(X, Y)$  instead of  $\text{Ext}^1(X, Y)$ . Recall that a module  $X$  is said to be *exceptional* provided it is indecomposable and has no self-extensions (this means that  $\text{Ext}(X, X) = 0$ ).

Following Roiter [Ro], we say that a module  $M$  is *normal* provided there is no proper direct decomposition  $M = M' \oplus M''$  such that  $M'$  generates  $M''$  (this means: if  $M = M' \oplus M''$  and  $M'$  generates  $M''$ , then  $M'' = 0$ ). Of course, given a module  $M$ , there is a direct decomposition  $M = M' \oplus M''$  such that  $M'$  is normal and  $M'$  generates  $M''$  and one can show that  $M'$  is determined by  $M$  uniquely up to isomorphism, thus we call  $M' = \nu(M)$  a *normalization* of  $M$ . This was shown already by Roiter [Ro], and later by Auslander-Smalø [AS]. It is also a consequence of the following Lemma which will be needed for our further considerations.

**Lemma 1.** (a) *Let  $(f_1, \dots, f_t, g): X \rightarrow X^t \oplus Y$  be an injective map for some natural number  $t$ , with all the maps  $f_i$  in the radical of  $\text{End}(X)$ . Then  $X$  is cogenerated by  $Y$ .*

(b) *Let  $(f_1, \dots, f_t, g): X^t \oplus Y \rightarrow X$  be a surjective map for some natural number  $t$ , with all the maps  $f_i$  in the radical of  $\text{End}(X)$ , then  $Y$  generates  $X$ .*

Proof. (a) Assume that the radical  $J$  of  $\text{End}(X)$  satisfies  $J^m = 0$ . Let  $W$  be the set of all compositions  $w$  of at most  $m - 1$  maps of the form  $f_i$  with  $1 \leq i \leq t$  (including  $w = 1_X$ ). We claim that  $(gw)_{w \in W}: X \rightarrow Y^{|W|}$  is injective. Take a non-zero element  $x$  in  $X$ . Then there is  $w \in W$  such that  $w(x) \neq 0$  and  $f_i w(x) = 0$  for  $1 \leq i \leq t$ . Since  $(f_1, \dots, f_t, g)$  is injective and  $w(x) \neq 0$ , we have  $(f_1, \dots, f_t, g)(w(x)) \neq 0$ . But  $f_i w(x) = 0$  for  $1 \leq i \leq t$ , thus  $g(w(x)) \neq 0$ . This completes the proof.  $\square$

(b) This follows by duality.  $\square$

**Corollary (Uniqueness of normalization).** *Let  $M$  be a module. Assume that  $M = M_0 \oplus M_1 = M'_0 \oplus M'_1$  such that both  $M_0$  and  $M'_0$  generate  $M$ . Then there is a module  $N$  which is a direct summand of both  $M_0$  and  $M'_0$  which generates  $M$ .*

Proof: We may assume that  $M$  is multiplicity free. Write  $M_0 \simeq N \oplus C$ ,  $M'_0 \simeq N \oplus C'$ , such that  $C, C'$  have no indecomposable direct summand in common. Now,  $N \oplus C$  generates  $N \oplus C'$  generates  $N \oplus C$  generates  $C$ . We see that  $N \oplus C$  generates  $C$ , such that the maps  $C \rightarrow C$  used belong to the radical of  $\text{End}(C)$  (since they factor through  $\text{add}(N \oplus C')$  and no indecomposable direct summand of  $C$  belongs to  $\text{add}(N \oplus C')$ ). Lemma 1 asserts that  $N$  generates  $C$ , thus it generates  $M$ .  $\square$

**Proposition 1.** *If  $T$  has no self-extensions, then  $T$  is a cover for the torsion class  $\mathcal{T}(T)$ . Conversely, if  $\mathcal{T}$  is a torsion class with cover  $C$ , then  $\nu(C)$  has no self-extensions.*

Proof. For the first assertion, one has to observe that  $\mathcal{G}(T)$  is closed under extensions, thus equal to  $\mathcal{T}(T)$ . This is a standard result say in tilting theory. Here is the argument: let  $g': T' \rightarrow M'$  and  $g'': T'' \rightarrow M''$  be surjective maps with  $T', T''$  in  $\text{add } T$ . Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence. The induced exact sequence with respect to  $g''$  is of the form  $0 \rightarrow M' \rightarrow Y_1 \rightarrow T'' \rightarrow 0$  with a surjective map  $g_1: Y_1 \rightarrow M$ . Since  $\Lambda$  is hereditary and  $g'$  is surjective, there is an exact sequence  $0 \rightarrow T' \rightarrow Y_2 \rightarrow T'' \rightarrow 0$

with a surjective map  $g_2: Y_2 \rightarrow Y_1$ . Since  $\text{Ext}(T'', T') = 0$ , we see that  $Y_2$  is isomorphic to  $T' \oplus T''$ , thus in  $\text{add } T$ . And there is the surjective map  $g_1 g_2: Y_2 \rightarrow M$ .

For the converse, we may assume that  $C$  is normal and have to show that  $C$  has no self-extension. Let  $C_1, C_2$  be indecomposable direct summands of  $C$  and assume for the contrary that there is a non-split exact sequence

$$0 \rightarrow C_1 \rightarrow M \rightarrow C_2 \rightarrow 0.$$

Now  $M$  belongs to  $\mathcal{T}$ , thus it is generated by  $C$ , say there is a surjective map  $C' \rightarrow M$  with  $C' \in \text{add } C$ . Write  $C' = C_2^t \oplus C''$  such that  $C_2$  is not a direct summand of  $C''$ . Consider the surjective map  $C_2^t \oplus C'' \rightarrow M \rightarrow C_2$ . Since the last map  $M \rightarrow C_2$  is not a split epimorphism, all the maps  $C_2 \rightarrow C_2$  involved belong to the radical of  $\text{End}(C_2)$ . According to Lemma 1,  $C''$  generates  $C_2$ . This contradicts the assumption that  $C$  is normal.  $\square$

**Remark.** As we have mentioned, normal modules have been considered by Roiter, but actually, he used a slightly deviating name, calling them "normally indecomposable".

## 2. Ext-cycles.

An *Ext-cycle* of cardinality  $t$  is a sequence  $X_1, X_2, \dots, X_m$  of pairwise orthogonal bricks such that  $\text{Ext}(X_{i-1}, X_i) \neq 0$  for  $1 \leq i \leq m$ , with  $X_0 = X_m$ . An *Ext-pair* is an Ext-cycle of cardinality 2 consisting of exceptional modules. (One may call an Ext-cycle  $X_1, \dots, X_m$  *minimal* provided there is no Ext-cycle of smaller cardinality which uses (some of) these modules. Using this definition, the Ext-pairs are just the minimal Ext-cycles of cardinality 2.)

**Proposition 2.** *If  $X_1, X_2, \dots, X_m$  is an Ext-cycle, then  $\mathcal{T}(X_1, \dots, X_m)$  has no cover.*

Proof: Let  $\mathcal{F} = \mathcal{F}(X_1, \dots, X_m)$  be the extension closure of  $X_1, \dots, X_m$ , thus the class of modules with a filtration with factors of the form  $X_i$ , where  $1 \leq i \leq m$ . According to [R],  $\mathcal{F}$  is an abelian subcategory with exact embedding functor, with (relative) simple objects the modules  $X_1, \dots, X_m$ . The objects in  $\mathcal{F}$  have finite (relative) length, thus also the (relative) Loewy length for these objects is defined. We denote by  $\mathcal{F}_t$  the full subcategory of objects in  $\mathcal{F}$  of (relative) Loewy length at most  $t$ .

We have

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_t \subseteq \dots$$

and therefore

$$\mathcal{G}(\mathcal{F}_1) \subseteq \mathcal{G}(\mathcal{F}_2) \subseteq \dots \subseteq \mathcal{G}(\mathcal{F}_t) \subseteq \dots,$$

Let  $\mathcal{G} = \bigcup_t \mathcal{G}(\mathcal{F}_t)$ . We claim that  $\mathcal{G} = \mathcal{T}(X_1, \dots, X_m)$ . The modules in  $\mathcal{G}$  belong to  $\mathcal{T}(X_1, \dots, X_m)$  and  $X_1, \dots, X_m$  belong to  $\mathcal{G}$ . Thus, it is sufficient to show that  $\mathcal{G}$  is a torsion class.

Since  $\mathcal{G}$  is the filtered union of classes closed under epimorphisms, it is closed under epimorphisms. In order to show that  $\mathcal{G}$  is closed under extensions, we follow the proof for the first assertion of Proposition 1 as closely as possible: Let  $g': F' \rightarrow M'$  and  $g'': F'' \rightarrow M''$  be surjective maps with  $F', F''$  in  $\text{add } \mathcal{F}_s$  for some  $s$ . Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence. The induced exact sequence with respect to  $g''$  is of the form  $0 \rightarrow M' \rightarrow Y_1 \rightarrow F'' \rightarrow 0$  with a surjective map  $g_1: Y_1 \rightarrow M$ . Since  $\Lambda$  is hereditary and

$g'$  is surjective, there is an exact sequence  $0 \rightarrow F' \rightarrow Y_2 \rightarrow F'' \rightarrow 0$  with a surjective map  $g_2: Y_2 \rightarrow Y_1$ . Since  $F', F''$  belong to  $\mathcal{F}$  and their (relative) Loewy length is at most  $s$ , the exact sequence shows that  $M$  also belongs to  $\mathcal{F}$  and has (relative) Loewy length at most  $2s$ . The surjective map  $g_1 g_2: Y_2 \rightarrow M$  shows that  $M$  is in  $\mathcal{G}(\mathcal{F}_{2s}) \subseteq \mathcal{G}$ .

Now assume that  $C$  is a cover for  $\mathcal{G}$ . The module  $C$  belongs to  $\mathcal{G}(\mathcal{F}_r)$  for some  $r$ , thus there is an epimorphism  $f: F \rightarrow C$  for some  $F \in \mathcal{F}_r$ . With  $C$  also  $F$  is a cover for  $\mathcal{G}$ . Note that there is a module  $F'$  which belongs to  $\mathcal{F}_{r+1}$  and not to  $\mathcal{F}_r$ , for example any object in  $\mathcal{F}$  which is (relative) serial and has (relative) length equal to  $r+1$ . Since  $F'$  is in  $\mathcal{G}$ , and  $F$  is a cover of  $\mathcal{G}$ , the module  $F'$  is generated by  $F$ . But if  $F'$  is generated by  $F$ , its (relative) Loewy length is at most  $r$ . This means that  $F'$  is in  $\mathcal{F}_r$ , a contradiction.  $\square$

### 3. Construction of Ext-pairs.

**Proposition 3.** *A connected hereditary artin algebra which is representation-infinite and has at least three simple modules has Ext-pairs.*

Given a finite dimensional algebra  $R$ , we denote by  $Q(R)$  its Ext-*quiver*: its vertices are the isomorphism classes  $[S]$  of the simple  $R$ -modules  $S$ , and given two simple  $R$ -modules  $S, S'$ , there is an arrow  $[S] \rightarrow [S']$  provided  $\text{Ext}^1(S, S') \neq 0$ . If  $R$  is hereditary, then clearly  $Q(R)$  is directed. If necessary, we endow  $Q(R)$  with a valuation as follows: Given an arrow  $S \rightarrow S'$ , consider  $\text{Ext}(S, S')$  as a left  $\text{End}(S)^{\text{op}}$ -module or as a left  $\text{End}(S')$ -module and put

$$v([S], [S']) = (\dim_{\text{End}(S)} \text{Ext}(S, S'))(\dim_{\text{End}(S')^{\text{op}}} \text{Ext}(S, S'))$$

(note that in contrast to [DR], we only will need the product of the two dimensions, not the pair). Given a vertex  $i$  of  $Q(R)$ , we denote by  $S(i), P(i), I(i)$  a simple, projective or injective module corresponding to the vertex  $i$ , respectively.

We later will use the following: If  $Q(\Lambda) = (1 \rightarrow 2)$ , then the arrow  $1 \rightarrow 2$  has valuation at least 2 if and only if  $I(2)$  is not projective if and only if  $P(1)$  is not injective; if the arrow  $1 \rightarrow 2$  has valuation at least 3, then  $\tau S(1)$  (where  $\tau$  is the Auslander-Reiten translation) is neither projective, nor a neighbor of  $P(1)$  in the Auslander-Reiten quiver, consequently  $\text{Hom}(P(1), \tau^2 S(1)) \neq 0$ , thus  $\text{Ext}(\tau S(1), P(1)) \neq 0$ .

For any hereditary algebra  $\Lambda$  with  $Q(\Lambda)$  being a tree quiver, it is easy to construct a sincere exceptional module, using induction: If  $Q'$  is a subquiver of  $Q$  such that  $Q$  is obtained from  $Q'$  by adding just one vertex  $\omega$  and one arrow, and  $M'$  is an exceptional module for the restriction of  $\Lambda$  to  $Q'$ , then let  $M$  be the universal extension of  $M'$  by copies of  $S(\omega)$ ; here we consider extensions from above or from below, provided  $\omega$  is a source or a sink, respectively.

For the proof of Proposition 3, we consider four special cases:

#### Case 1. The algebra $\Lambda$ is tame.

We use the structure of the Auslander-Reiten quiver of  $\Lambda$  as presented in [DR]. Since we assume that  $\Lambda$  has at least 3 vertices, there is a tube of rank  $r \geq 2$ . The simple regular modules in this component form an Ext-cycle of cardinality  $r$ , say  $X_1, \dots, X_r$ . There is a unique indecomposable module  $Y$  with a filtration  $Y = Y_0 \supset Y_1 \supset \dots \supset Y_{r-1} = 0$  such that  $Y_{i-1}/Y_i = X_i$  for  $1 \leq i \leq r-1$ . Clearly, the pair  $Y, X_r$  is an Ext-pair.

#### Case 2. The quiver $Q = Q(\Lambda)$ is not a tree.

Deleting, if necessary, vertices, we may assume that the underlying graph of  $Q$  is a cycle. Let  $w$  be a path from a sink  $i$  to a source  $j$  of smallest length, let  $Q'$  be the subquiver of  $Q$  given by the vertices and the arrows which occur in  $w$ . Not every vertex of  $Q$  belongs to  $Q'$ , since otherwise  $Q$  is obtained from  $Q'$  by adding just arrows, thus by adding a unique arrow, namely an arrow  $i \rightarrow j$ . But then this arrow is also a path from a sink to a source, and it has length 1. By the minimality of  $w$ , we see that also  $w$  has length 1 and therefore  $Q$  has just the two vertices  $i, j$ . But then  $Q$  can have only one arrow, thus is a tree. This is a contradiction.

Let  $Q''$  be the full subquiver given by all vertices of  $Q$  which do not belong to  $Q'$ . Of course,  $Q''$  is connected (it is a quiver of type A). Let  $X$  be an exceptional module with support  $Q'$  and  $Y$  an exceptional module with support  $Q''$ . Since  $Q', Q''$  have no vertex in common, we see that  $\text{Hom}(X, Y) = 0 = \text{Hom}(Y, X)$ .

There is an arrow  $i \rightarrow j''$  with  $j''$  a vertex of  $Q''$ . This arrow shows that  $\text{Ext}^1(X, Y) \neq 0$ . Similarly, there is an arrow  $i'' \rightarrow j$  with  $i''$  a vertex of  $Q''$ . This arrow shows that  $\text{Ext}^1(Y, X) \neq 0$ .

We consider now algebras  $\Lambda$  with Ext-quiver  $1 \rightarrow 2 \rightarrow 3$ . We denote by  $\Lambda'$  the restriction of  $\Lambda$  to the subquiver with vertices 1, 2, and by  $\Lambda''$  the restriction of  $\Lambda$  to the subquiver with vertices 2, 3. Given a representation  $M$ , let  $M_3$  be the sum of all submodules of  $M$  which are isomorphic to  $S(3)$ , then  $M/M_3$  is a  $\Lambda'$ -module.

**Lemma 2.** *Let  $X, Y$  be  $\Lambda$ -modules. If  $X_3 = 0$  and  $\text{Ext}^1(Y/Y_3, X) \neq 0$ , then also  $\text{Ext}^1(Y, X) \neq 0$ .*

Proof: The exact sequence  $0 \rightarrow Y_3 \rightarrow Y \rightarrow Y/Y_3 \rightarrow 0$  yields an exact sequence

$$\text{Hom}(Y_3, X) \rightarrow \text{Ext}^1(Y/Y_3, X) \rightarrow \text{Ext}^1(Y, X)$$

The first term is zero, since  $Y_3$  is a sum of copies of  $S(3)$  and  $X_3 = 0$ . Thus, the map  $\text{Ext}^1(Y/Y_3, X) \rightarrow \text{Ext}^1(Y, X)$  is injective.

**Case 3.**  $Q(\Lambda) = (1 \rightarrow 2 \rightarrow 3)$ , and  $v(1, 2) \geq 2$ ,  $v(2, 3) \geq 2$ .

Let  $X = S(2)$  and let  $Y$  be the universal extension of  $X$  using the modules (1) and  $S(3)$  (thus, we form the universal extension from above using copies of  $S(1)$  and the universal extension from below using copies of  $S(3)$ ). Clearly,  $Y$  is exceptional. Since the socle of  $Y$  consists of copies of  $S(3)$ , we have  $\text{Hom}(S(2), Y) = 0$ . Since the top of  $Y$  consists of copies of  $S(1)$ , we have  $\text{Hom}(Y, S(2)) = 0$ .

Since  $v(1, 2) \geq 2$ , the module  $Y/Y_3$  is not a projective  $\Lambda'$ -module. As a consequence,  $\text{Ext}(Y/Y_3, S(2)) \neq 0$ . Lemma 2 shows that also  $\text{Ext}(Y, S(2)) \neq 0$ . By duality, we similarly see that  $\text{Ext}(S(2), Y) \neq 0$ .

**Case 4.**  $Q(\Lambda) = (1 \rightarrow 2 \rightarrow 3)$ , and  $v(1, 2) \geq 3$ ,  $v(2, 3) = 1$ .

Let  $X = P(1)/P(1)_3$  (thus  $X$  is the projective  $\Lambda'$ -module with top  $S(1)$ ). Let  $Y = \tau X$ , where  $\tau = D \text{Tr}$  is the Auslander-Reiten translation in  $\text{mod } \Lambda$ . Of course, both modules  $X, Y$  are exceptional. Since  $Y = \tau X$ , we know already that  $\text{Ext}^1(X, Y) \neq 0$ .

We claim that  $Y/Y_3 = \tau' S(1)$ , where  $\tau'$  is the Auslander-Reiten translation of  $\Lambda'$ . Since  $P(1)_3 = S(3)^a$  for some  $a \geq 1$ , a minimal projective presentation of  $X$  has the form

$$(*) \quad 0 \rightarrow S(3)^a \rightarrow P(1) \rightarrow X \rightarrow 0,$$

thus the defining exact sequences for  $Y = \tau X$  is of the form

$$0 \rightarrow Y \rightarrow I(3)^a \rightarrow S(1) \rightarrow 0.$$

In order to obtain  $\tau' S(1)$ , we start with a minimal projective presentation

$$(**) \quad 0 \rightarrow S(2)^a \rightarrow P'(1) \rightarrow S(1) \rightarrow 0,$$

where  $P'(1)$  is the projective cover of  $S(1)$  as a  $\Lambda'$ -module (actually,  $P'(1) = X$ ). Since  $\nu(2, 3) = 1$ , the number  $a$  in  $(*)$  and  $(**)$  is the same. The defining exact sequences for  $Y = \tau X$  and  $\tau' S(1)$  are part of the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \tau' S(1) & \longrightarrow & I(2)^a & \longrightarrow & S(1) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \parallel \\
0 & \longrightarrow & Y & \longrightarrow & I(3)^a & \longrightarrow & S(1) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \\
& & S(3)^a & \xlongequal{\quad} & S(3)^a & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 
\end{array}$$

The left column shows that  $Y/Y_3 = \tau' S(1)$ .

We have noted already that  $v(1, 2) \geq 3$  implies that  $\text{Ext}(\tau' S(1), P'(1)) \neq 0$ . According to Lemma 2, we see that  $\text{Ext}(Y, X) \neq 0$ .

Finally, let us show that  $X, Y$  are orthogonal. Any homomorphism  $Y \rightarrow X$  vanishes on  $Y_3$ , since  $X$  has no composition factor  $S(3)$ . Now  $Y/Y_3$  is indecomposable and not projective as a  $\Lambda'$ -module, whereas  $X$  is a projective  $\Lambda'$ -module, thus  $\text{Hom}(Y, X) = \text{Hom}(Y/Y_3, X) = 0$ .

On the other hand, the restriction  $X''$  of  $X$  to the subquiver  $Q''$  with vertices 2, 3 is a sum of copies of  $S(2)$ , whereas the restriction of  $Y$  to the subquiver  $Q''$  is a projective-injective module. It follows that the restriction of any homomorphism  $f: X \rightarrow Y$  vanishes on  $X''$ . Thus  $f$  factors through a direct sum of copies of  $S(1)$ . But  $S(1)$  is injective and obviously not a submodule of  $Y$ . It follows that  $f = 0$ .

**Remark.** Concerning the cases 3 and 4, there is an alternative proof which uses dimension vectors and the Euler form on the Grothendieck group  $K_0(\Lambda)$ . But for this approach, one needs to deal with the valuation of  $Q(\Lambda)$  as in [DR], attaching to any arrow  $i \rightarrow j$  a pair  $(a, b)$  of positive numbers instead of the single number  $v(i, j) = ab$ .

**Proof of Proposition 3.** Let  $\Lambda$  be connected, hereditary, representation-infinite, with at least 3 simple modules. Case 2 shows that we can assume that  $Q(\Lambda)$  is a tree. Assume that there is a subquiver  $Q'$  such that at least two of the arrows have valuation at least 2,

choose such a  $Q'$  of minimal length. We want to construct an Ext-pair for the restriction of  $\Lambda$  to  $Q'$ . Using reflection functors (see [DR]), we can assume that  $Q'$  has orientation  $1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n$ . If  $n = 3$ , then this is case 3. Thus assume  $n \geq 4$ . The minimality of  $Q'$  asserts that  $\nu(i, i+1) = 1$  for  $2 \leq i \leq n-2$ . If we denote by  $\Lambda'$  the restriction of  $\Lambda$  to  $Q'$ , then  $\Lambda'$  has a full exact abelian subcategory  $\mathcal{U}$  which is equivalent to the module category of an algebra as discussed in case 3 (namely the subcategory of all  $\Lambda'$ -modules which do not have submodules of the form  $S(i)$  with  $2 \leq i \leq n-2$  and no factor modules of the form  $S(i)$  with  $3 \leq i \leq n-1$ ). Since  $\mathcal{U}$  has Ext-pairs, also  $\text{mod } \Lambda$  has Ext-pairs. Thus, we can assume that at most one arrow  $i \rightarrow j$  has valuation greater than 2. If  $\nu(i, j) \geq 3$ , then we take a connected subquiver  $Q'$  with 3 vertices containing this arrow  $i \rightarrow j$ . If necessary, we use again reflection functors in order to change the orientation so that we are in case 4. Thus we are left with the representation-infinite algebras  $\Lambda$  with the following properties:  $Q(\Lambda)$  is a tree, there is no arrow with valuation greater than 2 and at most one arrow with valuation equal to 2. It is easy to see that  $Q(\Lambda)$  contains a subquiver  $Q'$  such that the restriction of  $\Lambda$  to  $Q'$  is tame, thus we can use case 1.  $\square$

**Proof of Theorem.** Let  $\Lambda$  be connected and hereditary. If  $\Lambda$  is representation-finite, then  $\text{tors } \Lambda = \text{f-tors } \Lambda$ , thus  $\text{f-tors } \Lambda$  is a lattice. If  $\Lambda$  has precisely two simple modules, then  $\text{f-tors } \Lambda$  can be described easily (see the proof of Proposition 2.2 in [IRTT] which works in general), it obviously is a lattice.

On the other hand, if  $\Lambda$  is representation-infinite and has at least three simple modules, then Proposition 3 asserts that  $\Lambda$  has an Ext-pair, say  $X, Y$ . Since  $X, Y$  are exceptional modules, Proposition 1 shows that  $\mathcal{T}(X) = \mathcal{G}(X)$  and  $\mathcal{T}(Y) = \mathcal{G}(Y)$  both belong to  $\text{f-tors } \Lambda$ . The join of  $\mathcal{T}(X)$  and  $\mathcal{T}(Y)$  in  $\text{tors } \Lambda$  is  $\mathcal{T}(X, Y)$ . According to Proposition 2,  $\mathcal{T}(X, Y)$  does not belong to  $\text{f-tors } \Lambda$ .  $\square$

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